Dominator Colorings and Safe Clique Partitions

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ABSTRACT

Given a graph G, the dominator coloring problem seeks a proper coloring of G with the additional property that every vertex in the graph dominates an entire color class. The safe clique partition problem seeks a partition of the vertices of a graph into cliques with the additional property that for each vertex v, there is a clique that has no element in the open neighborhood of v. We typically seek to minimize the number of color classes or cliques used, respectively. In this paper, we study these two problems and consider the relationship between them.

Key Words: coloring, domination, clique partition.

AMS Subject Classification: 05C15, 05C69.

1 Introduction and motivation

A *dominating set* S is a subset of the vertices in a graph such that every vertex in the graph either belongs to S or has a neighbor in S. The topic has long been of interest to researchers [11, 12]. The associated decision problem, DOMINATING SET, occupies a prominent place in the computational complexity literature [9]. This definition also leads naturally to the associated optimization problem, which is to find a dominating set of minimum cardinality. Numerous variants of this problem have been studied [1, 11, 12, 17]. DOMINATING SET is NP-complete on arbitrary graphs [9]. It is also NP-complete on several classes of graphs, including planar graphs [9], bipartite graphs [6], and chordal graphs [3]. The problem can be solved in polynomial time on, for example, AT-free graphs [16], permutation graphs [8], interval graphs [2], and trees [9].

A *proper coloring* of a graph $G = (V(G), E(G))$ is a function from the vertices of the graph to a set of *colors* such that any two adjacent vertices have different colors. Graph coloring is used as a model for a vast number of practical problems involving allocation of scarce resources (e.g., scheduling problems), and has played a key role in the development of graph theory and, more generally, discrete mathematics and combinatorial optimization. Graph k-colorability is NP-complete in the general case, although the problem is solvable in polynomial time for many classes [9]. For a comprehensive treatment of classical and modern results, and a relatively current view of the frontier, see Jensen and Toft [15].

A graph has a *dominator coloring* if it has a proper coloring in which each vertex of the graph dominates every vertex of some color class. The *dominator chromatic number* $\chi_d(G)$ is the minimum number of color classes in a dominator coloring of a graph G. A $\chi_d(G)$-coloring of G is any dominator coloring with $\chi_d(G)$ colors. Our study of this problem is motivated by [4] and [13].

We start with notation and more formal problem definitions. Let $G = (V(G), E(G))$ be a graph with $n = |V(G)|$ and $m = |E(G)|$. For any vertex $v \in V(G)$, the *open neighborhood* of v is the set $N(v) = \{u \mid uv \in E(G)\}$ and the *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. Similarly, for any set $S \subseteq V(G)$, $N(S) = \cup_{v \in S} N(v) - S$ and $N[S] = N(S) \cup S$. A set S is a *dominating set* if $N[S] = V(G)$. The minimum cardinality of a dominating set of G is denoted by $\gamma(G)$. The *distance*, $d(u, v)$, between two vertices u and v in G is the smallest number of edges on a path between u and v in G. The *eccentricity*, $e(v)$, of a vertex v is the largest distance from v to any vertex of G. The *radius*, $rad(G)$, of G is the smallest eccentricity in G. The *diameter*, $diam(G)$, of G is the largest eccentricity in G.

A *graph coloring* is a mapping $f : V(G) \rightarrow C$, where C is a set of colors (frequently $C \subseteq \mathbf{Z}^+$). A coloring f is *proper* if, for all $x, y \in V(G)$, $x \in N(y)$ implies $f(x) \neq f(y)$. A *k-coloring* of G is a coloring that uses at most k colors. The *chromatic number* of G is $\chi(G) = \min\{k | G \text{ has a proper } k\text{-coloring}\}$. A coloring of G can also be thought of as a partition of $V(G)$ into color classes $V_1, V_2, ..., V_q$, and a proper coloring of G is then a coloring in which each V_i, $1 \leq i \leq q$ is an independent set of G, *i.e.,* for each i, the subgraph of G

induced by V_i contains no edges.

Our aim in this paper is to introduce and study the dominator coloring problem. Below, we address the computational complexity of the problem in general. In Section 2, we give some preliminary results. In Section 3, we consider the problem on paths and caterpillars. Section 4 introduces a related problem that can be used to solve the dominator coloring problem in some cases. Section 5 provides a brief discussion of the results and some directions for further research.

1.1 Complexity

In this section we formally establish the difficulty of finding the dominator coloring number of an arbitrary graph. First we define some relevant decision problems.
CHROMATIC NUMBER Given a graph G and a positive integer k, does there exist a function $f : V(G) \to \{1, 2, ..., k\}$ such that $f(u) \neq f(v)$ whenever $\{u, v\} \in E(G)$?
DOMINATOR CHROMATIC NUMBER Given a graph G and a positive integer k, does there exist a function $f : V(G) \to \{1, 2, ..., k\}$ such that $f(u) \neq f(v)$ whenever $\{u, v\} \in E(G)$ and for all $v \in V(G)$ there exists a color i such that $\{u \in V(G) : f(u) = i\} \subseteq N[v]$?

Theorem 1.1 DOMINATOR CHROMATIC NUMBER *is NP-complete.*

Proof. DOMINATOR CHROMATIC NUMBER is clearly in NP since we can efficiently verify that an assignment of colors to the vertices of G is both a proper coloring and that every vertex dominates some color class.

Now we transform CHROMATIC NUMBER to DOMINATOR CHROMATIC NUMBER. Consider an arbitrary instance (G, k) of CHROMATIC NUMBER. Create an instance (G', k') of DOMINATOR CHROMATIC NUMBER as follows. Add a vertex v' to G and add an edge from v' to every vertex in G. Formally, $V(G') = v' \cup V(G)$ and $E(G') = \cup_{v \in V(G)}\{v', v\} \cup E(G)$. Set $k' \leftarrow k + 1$.

Suppose G has a proper coloring using k colors. Then the coloring of G' that colors v' with a new color and otherwise retains the coloring from G is a proper coloring of G'. Since $v' \subseteq N[u]$ for every $u \in V(G')$, this coloring is a dominator coloring, and is uses $k' = k + 1$ colors.

Now suppose G' has a dominator coloring using k' colors. Since v' is adjacent to every other vertex in G', it must be the only vertex of its color in the hypothesized coloring. Then the removal of v' leaves a proper coloring of G that uses $k' - 1 = k$ colors. □

2 Preliminary results

Let G be a connected graph of order $n \geq 2$. The coloring that assigns a different color to each vertex of G is a proper coloring with each vertex dominating the color class that contains it.

Thus the dominator chromatic number is defined for every graph, and $\chi_d(G) \leq n$.

We first consider the dominator chromatic number for some simple cases. It is easy to observe the following.

Observation 2.1 *The star $K_{1,n}$ and the complete graph K_n have $\chi_d(K_{1,n}) = 2$ and $\chi_d(K_n) = n$, respectively.*

To further illustrate the concept, we will consider the problem on the *double star*. A double star $S_{a,b}$ is a tree of diameter 3. Let $S_{a,b}$ be the double star with central vertices u and v, with $\deg u = a \geq 2$ and $\deg v = b \geq 2$. Let $X = \{x_1, x_2 \ldots, x_{a-1}\}$, $Y = \{y_1, y_2 \ldots, y_{b-1}\}$, $N(v) = \{u\} \cup Y$ and $N(u) = \{v\} \cup X$. See Figure 1 below.

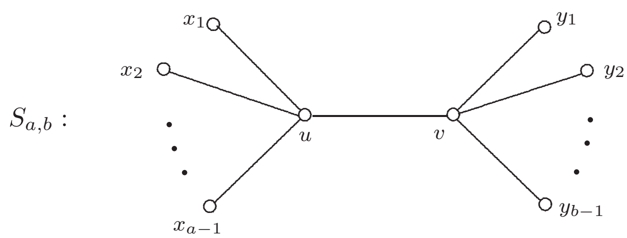

Figure 1: The graph $S_{a,b}$

Now we determine $\chi_d(S_{a,b})$.

Lemma 2.2 *For $a + b \geq 4$, $\chi_d(S_{a,b}) = 3$.*

Proof. Consider a proper coloring of $S_{a,b}$ in which $V_1 = X \cup Y$, $V_2 = \{u\}$, and $V_3 = \{v\}$. Then each vertex in the set $\{u\} \cup X$ dominates the color class V_2, and each vertex in the set $\{v\} \cup Y$ dominates the color class V_3. Therefore this is a dominator coloring and $\chi_d(S_{a,b}) \leq 3$. Now consider a different proper coloring in which $V_1 = X \cup \{v\}$ and $V_2 = Y \cup \{u\}$. This is not a dominator coloring since, for example, x_1 does not dominate a color class. But this is the only proper coloring of $S_{a,b}$ using two colors. Thus $\chi_d(S_{a,b}) > 2$, establishing the result. $\square$

We are now prepared to determine bounds on the dominator chromatic number for arbitrary graphs. Since a dominator coloring has to be a proper coloring, the following is immediate.

Observation 2.3 *For any graph G,*

$$\chi(G) \leq \chi_d(G).$$

Strict inequality as well as equality in Observation 2.3 is possible. As we saw in the proof of Lemma 2.2, $2 = \chi(S_{a,b}) < \chi_d(S_{a,b}) = 3$ when $a + b \geq 4$. But $\chi(K_{1,n}) = \chi_d(K_{1,n}) = 2$ and so the bound in Observation 2.3 is sharp.

Now we present an upper bound for the dominator chromatic number in terms of the independence number $\alpha(G)$ of a graph G, followed by a realization result. Recall that the *independence number* $\alpha(G)$ is the order of a largest set of independent vertices.

Proposition 2.4 *For a connected graph G of order $n \geq 3$,*

$$\chi_d(G) \leq n + 1 - \alpha(G),$$

and this bound is sharp.

Proof. Let G be a connected graph of order $n \geq 3$. Let I be a maximum independent set of G, and let $V(G) - I = \{v_1, v_2, \ldots, v_{n-\alpha(G)}\}$. Then define $V_1, V_2, ..., V_{n-\alpha(G)+1}$ to be the coloring of G in which $V_j = \{v_j\}$ $(1 \leq j \leq n - \alpha(G))$, and $V_{n-\alpha(G)+1} = I$. Since G is connected, and I is independent, it follows that every vertex in I is adjacent to some vertex v_j $(1 \leq j \leq n - \alpha(G))$, so it dominates the color class V_j, and each vertex v_j dominates the color class V_j. Therefore $V_1, V_2, ..., V_{n-\alpha(G)+1}$ is a dominator coloring with colors $1, 2, ..., n - \alpha(G) + 1$, so $\chi_d(G) \leq n + 1 - \alpha(G)$. Also note that since $\alpha(S_{a,b}) = n - 2$ and $\chi_d(S_{a,b}) = 3 = n + 1 - \alpha(S_{a,b})$, the bound from Proposition 2.4 is sharp. □

Observation 2.5 *For a connected graph G of order $n \geq 2$, we have that*

$$2 \leq \chi_d(G) \leq n,$$

and these bounds are sharp.

Sharpness is clear from $\chi_d(K_{1,n}) = 2$ and $\chi_d(K_n) = n$. Note also that there is a connected graph of order n with dominator chromatic number k for all pairs (n, k) with $n \geq 3$, $k \geq 2$, and $n \geq k$. To see this, we construct a graph $G_{k,n}$ with dominator chromatic number k and order $n \geq 3$. We obtain $G_{k,n}$ from the complete graph $K_k : v_1, v_2, \ldots, v_k$, by adding $n-k$ new vertices $u_1, u_2, \ldots, u_{n-k}$ together with $n-k$ new edges v_1u_i $(1 \leq i \leq n-k)$. Then the coloring $V_1, V_2, ..., V_k$ in which $V_j = \{v_j\}$ $(1 \leq j \leq k-1, j \neq 2)$, and $V_2 = \{v_2, u_1, u_2, \ldots, u_{n-k}\}$ is a minimum dominator coloring. Thus $\chi_d(G) = k$ and G has order n. See Figure 2.

Heretofore we have considered only connected graphs. If G is not connected with connected components $G_1, G_2, ..., G_k$, then we have the following result.

Proposition 2.6 *If G is a disconnected graph with components $G_1, G_2, \ldots, G_k$ with $k \geq 2$, then*

$$\max_{i \in \{1,2,\ldots,k\}} \chi_d(G_i) + k - 1 \leq \chi_d(G) \leq \sum_{i=1}^{k} \chi_d(G_i),$$

and these bounds are sharp.

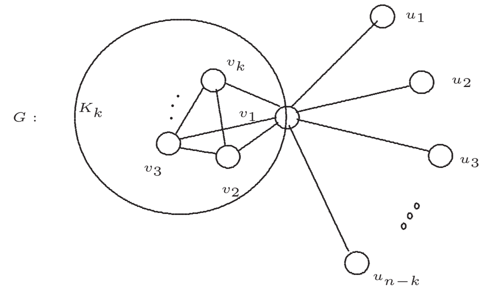

Figure 2: Graph $G_{k,n}$ of order n with $\chi_d(G_{k,n}) = k$.

Proof. For each i with $1 \leq i \leq k$, let $V_{i,1}, V_{i,2} \ldots, V_{i,q_i}$ be a dominator coloring of the component G_i. Then $\bigcup_{i=1}^{k} \{V_{i,1}, V_{i,2} \ldots, V_{i,q_i}\}$ is a dominator coloring of G, and $\chi_d(G) \leq \sum_{i=1}^{k} \chi_d(G_i)$.

Next, we show the lower bound. Let G_j be a component of G of maximum dominator chromatic number, *i.e.* $\chi_d(G_j) = \max_{i\in\{1,2,\ldots,k\}} \chi_d(G_i)$. For each $i \neq j$, G_i needs at least one new color, since a vertex in G_i must dominate an entire color class. This establishes the desired result.

These bounds are sharp since the left inequality is satisfied at equality when each $G_i \cong K_{1,n}$ for $n \geq 2$. Color each component in accordance with Observation 2.1, reusing the color used for the leaf vertices for each component. The right inequality is similarly satisfied when each component is a single vertex K_1. In this case every vertex needs a new color. □

The final result of this section is perhaps something of a surprise. If $H \subseteq G$ it is clear that $\chi(H) \leq \chi(G)$ since any proper coloring of G will be proper for H. The following result shows this is not the case for dominator coloring.

Lemma 2.7 *There exists a graph G for which $\chi_d(H) > \chi_d(G)$ where H is a subgraph of G. There also exists a graph G for which $\chi_d(H) < \chi_d(G)$ where H is a subgraph of G.*

Proof. Let G_1 be the empty graph on n vertices, let G_2 be the star $K_{1,n-1}$, and let G_3 be the complete graph on n vertices. Clearly $G_1 \subseteq G_2 \subseteq G_3$. We have

$$\chi_d(G_1) = n > \chi_d(G_2) = 2, \text{ but } \chi_d(G_2) = 2 < \chi_d(G_3) = n.$$

□

In [14] the authors discuss a strategy for obtaining bounds and approximations for combinatorial problems on graphs where the instance graph is replaced with a subgraph, the problem is solved on the subgraph, and then the solution is used to approximate or bound the solution to the problem on the original instance. For instance, if H is a spanning subgraph of G, then $\gamma(H) \geq \gamma(G)$ since a dominating set in H is clearly a dominating set in G as well. Thus $\gamma(H)$ is an upper bound on $\gamma(G)$ that, depending on the structure of H, might be easier to compute than $\gamma(G)$. On the other hand, the inequality is reversed if we consider chromatic number. It is easy to see that $\chi(H) \leq \chi(G)$ since a proper coloring of G is also a proper coloring of H. Many other combinatorial problems are similar in the sense that a solution on a subgraph provides some kind of bound. Unfortunately, the dominator chromatic number problem does *not* have this property, as Lemma 2.7 establishes.

3 Dominator chromatic numbers of paths and caterpillars

In the previous section, dominator chromatic numbers were determined for stars, double stars, and complete graphs. In this section we consider paths and caterpillars.

Lemma 3.1 *There is a $\chi_d(G)$-coloring of P_n $(n \geq 5)$ in which the subcoloring induced on vertices $v_1, v_2, \ldots, v_{n-3}$ is a dominator coloring of P_{n-3}.*

Proof. Let $n \geq 5$ and let f be a $\chi_d(G)$-coloring of P_n. Suppose that the restriction of f to $v_1, v_2, \ldots, v_{n-3}$ is not a dominator coloring of P_{n-3}. Then it must be the case that both $f(v_{n-4})$ and $f(v_{n-3})$ are reused colors and that $f(v_{n-2})$ is a new color. If we interchange the colors on v_{n-3} and v_{n-2}, the resulting coloring induces a dominator coloring of the subpath $v_1, v_2, \ldots, v_{n-3}$. Moreover, by interchanging the colors on v_{n-1} and v_n if necessary, we obtain a $\chi_d(G)$-coloring of P_n with the stated property. □

Lemma 3.2 *If $n \geq 8$, then $\chi_d(P_n) \geq 2 + \lceil \frac{n}{3} \rceil$.*

Proof. For $n = 8$ through $n = 10$, it is not hard to verify that the inequality holds, and is in fact sharp. For $n > 10$, we proceed by induction. Assume that $\chi_d(P_{n-k}) \geq 2 + \lceil \frac{n-k}{3} \rceil$ for all $8 \leq n - k < n$. Let f be a $\chi_d(G)$-coloring of P_n. From Lemma 3.1, we may assume that the coloring induced on vertices $v_1, v_2, \ldots, v_{n-3}$ is a dominator coloring for P_{n-3}. By the induction hypothesis, at least $2 + \lceil \frac{n-3}{3} \rceil$ colors are used to color vertices v_1 through v_{n-3}. If f assigns a new color to v_n, we are done, so suppose that f assigns a used color, say i, to v_n. Since v_n cannot possibly dominate all vertices with color i, it must be that $f(v_{n-1})$ is a new color. Thus either $f(v_{n-1})$ or $f(v_n)$ is a new color, so f uses at least $2 + \lceil \frac{n-3}{3} \rceil + 1 = 2 + \lceil \frac{n}{3} \rceil$ colors, and thus $\chi_d(P_n) \geq 2 + \lceil \frac{n}{3} \rceil$. □

Theorem 3.3 *The path P_n of order $n \geq 2$ has*

$$\chi_d(P_n) = \begin{cases} 1 + \lceil \frac{n}{3} \rceil; & n = 2, 3, 4, 5, 7 \\ 2 + \lceil \frac{n}{3} \rceil; & \textit{otherwise.} \end{cases}$$

Proof. The verification of cases $2 \leq n \leq 7$ is straightforward. By Lemma 3.2, it suffices to show that $\chi_d(P_n) \leq 2 + \lceil \frac{n}{3} \rceil$. We construct a dominator coloring $f : V(P_n) \to \{1, 2, \ldots, \chi_d(P_n)\}$ using exactly $2 + \lceil \frac{n}{3} \rceil$ colors as follows: For each $i \equiv 1 \pmod 6$ and $i \equiv 3 \pmod 6$, let $f(v_i) = 1$. For each $i \equiv 4 \pmod 6$ and $i \equiv 0 \pmod 6$, let $f(v_i) = 2$. Let $f(v_{3i+2}) = 3 + i$ for each $0 \leq i \leq \lfloor \frac{n}{3} \rfloor$. If $n \equiv 0 \pmod 3$, the coloring is complete. If $n \equiv 1 \pmod 3$, then both $f(v_n)$ and $f(v_{n-1})$ are in $\{1, 2\}$, so $f(v_n)$ will not dominate any class. In this case redefine $f(v_n) := 1 + f(v_{n-2})$, which is the next available new color. Finally, if $n \equiv 2 \pmod 3$, we are done. Each vertex labeled 1 or 2 dominates some uniquely colored neighbor, and each vertex colored k for $3 \leq k \leq 2 + \lceil \frac{n}{3} \rceil$ dominates its own color class. In each case, precisely $2 + \lceil \frac{n}{3} \rceil$ colors are used. The result follows. $\square$

Caterpillars are another important class that we will consider next. A *caterpillar* is a tree with the additional property that the removal of all end vertices leaves a path. This path is called the *spine* of the caterpillar, and the vertices of the spine are called *vertebrae*.

Lemma 3.4 *If G is a caterpillar, there is a $\chi_d(G)$-coloring of G in which the colors assigned to the vertices of degree at least 3 are distinct.*

Proof. Let G be a caterpillar, and let $S = \{v_1, v_2, \ldots, v_r\}$ be the set of vertices of degree at least 3. Let $\{V_1, V_2, \ldots, V_{\chi_d(G)}\}$ be a χ_d-coloring of G in which the number of colors that are repeated on S is minimum. If no color is repeated, we are done, so assume that some color, say c, is used on vertebrae $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$, where $k \geq 2$ and $v_{i_j} \in S$ for each $1 \leq j \leq k$. Since each vertex in S has degree at least 3, it follows that each vertex v_{i_j} has a leaf neighbor x_j. Since x_j does not dominate V_c, the vertices x_j $(1 \leq j \leq k)$ must be assigned distinct colors. Update the coloring of G by interchanging, for each $1 \leq j \leq k$, the color at x_j with the color at v_{i_j}. The result is still a dominator coloring, since both x_j and v_{i_j} now dominate the color class $\{v_{i_j}\}$, and is still optimal, since the number of colors in use has not changed. But fewer colors are repeated on the spine, contradicting our hypothesis that the number of colors repeated on the spine was already minimum. $\square$

Theorem 3.5 *If G is a caterpillar in which the vertices of degree less than 3 are independent, and if the spine of G contains exactly r vertices of degree at least 3, then $\chi_d(G) = r + 1$.*

Proof. By Lemma 3.4, there is an optimal coloring in which r colors are used on the vertices of degree at least 3 in the spine of G. Since no leaf can use a color used on the spine, it follows that $\chi_d(G) \geq r + 1$. It now suffices to produce a dominator coloring using $r + 1$ colors. For this, simply color each vertex of degree at least 3 with a unique color. This uses r colors. All other vertices will be assigned color $r + 1$. Since the degree 1 and 2 vertices are independent, this coloring is proper. Since G is connected, each such vertex is adjacent to a vertex of degree at least 3 and therefore dominates a color class. $\square$

Note that caterpillars that have *no* vertices of degree 2 (a subclass of the class considered above) are precisely those that can form the tree portion of *Halin* graphs, the latter being minimally 3-connected graphs formed by the union of a tree with no degree 2 vertices and a cycle on the leaves of the tree. See [5, 7, 10].

From here, we can observe that if adjacencies occur among the vertices of degree 2, then the dominator chromatic number must increase, but we do not in general know how to compute $\chi_d(G)$ for such graphs.

4 Safe clique partitions

In this section we introduce a variant of the *minimum clique partition* problem and show its relationship to the dominator coloring problem. Given a graph $G = (V(G), E(G))$, a *clique partition* is a partition of $V(G)$ into subsets $V_1, V_2, ..., V_q$ so that every V_i $(1 \leq i \leq q)$ is a clique in G. The *clique partition number* $\overline{\chi}(G)$ is the minimum value of q over all such partitions. Let $\overline{G}$ denote the graph complement of $G = (V(G), E(G))$, so that $\overline{G} = (V(G), \overline{E}(G))$ where $\overline{E}(G) = \{uv : uv \notin E(G)\}$. Then a graph coloring in G is a clique partition in $\overline{G}$, since the cliques of $\overline{G}$ are precisely the independent sets of G.

4.1 Dominator colorings and safe clique partitions

The dominator coloring problem on a graph G can also be recast as a problem on $\overline{G}$. We will call this the *safe clique partition* problem. Given a graph G and a clique partition $V_1, V_2, ..., V_q$, a vertex v of G is *safe* if there is some i $(1 \leq i \leq q)$ such that $N(v) \cap V_i = \emptyset$. Otherwise it is *unsafe*. We call a partition into cliques *safe* if every vertex is safe. The *safe clique partition number* of a graph G is the smallest integer q for which $V_1, V_2, ..., V_q$ is a safe clique partition. We denote this number using $\overline{\chi_d}(G)$. The following observation is immediate, since all safe clique partitions are clique partitions.

Observation 4.1 $\overline{\chi_d}(G) \geq \overline{\chi}(G)$.

As we have seen, the proper coloring constraints of the dominator coloring problem correspond to a particular partition of V into cliques in $\overline{G}$, *i.e.*, in safe clique partition. The constraint in dominator coloring that requires each vertex to dominate some color class is equivalent to requiring that the partition is safe. To see this, consider a vertex of $\overline{G}$ that has an open neighborhood which includes a member of every clique in the partition. Such a vertex is by definition unsafe, and the same vertex in G fails to be adjacent to at least one member of every color class in G. Therefore it fails to dominate an entire color class in G. Recall that a vertex of G that is its own color class always satisfies the requirement to dominate some color class, since it dominates itself. Similarly, any single vertex of $\overline{G}$ that is a clique in a clique partition is safe since its open neighborhood cannot contain a member of its own clique. This leads to the following observation.

Observation 4.2 $\overline{\chi_d}(G) = \chi_d(\overline{G})$.

4.2 Computing $\overline{\chi_d}(G)$

Now we consider some classes of graphs for which safe clique partitions of minimum cardinality can be found in polynomial time. It has long been known that the minimum clique partition problem can be solved in polynomial time on triangle-free graphs by matching [9]. The idea is simple. First recall that a matching is a set of pairwise nonadjacent edges in a graph. Since triangle free graphs contain no triangles, they have no cliques larger than K_2. A maximum matching M^* can be found efficiently, so we can efficiently compute $\overline{\chi}(G) = n - |M^*|$ for such graphs. We follow with some easy observations.

Observation 4.3 *Given a triangle free graph G of order n, a maximum matching M^* corresponds to a minimum clique partition. If the latter happens to be safe, then $\overline{\chi_d}(G) = n - |M^*|$.*

Observation 4.4 *Consider P_n, the path on n vertices. For $n \geq 5$, $\overline{\chi_d}(P_n) = \overline{\chi}(P_n) = n - |M^*| = \lceil \frac{n}{2} \rceil$. Also, $\overline{\chi_d}(P_2) = 2$ and $\overline{\chi_d}(P_3) = \overline{\chi_d}(P_4) = 3$.*

Computing $\overline{\chi_d}(G)$ for cycles is very similar.

Observation 4.5 *Consider C_n, the cycle on n vertices. For $n \geq 5$, $\overline{\chi_d}(C_n) = \overline{\chi}(C_n) = n - |M^*| = \lceil \frac{n}{2} \rceil$. Also, $\overline{\chi_d}(C_3) = \overline{\chi_d}(C_4) = 3$.*

4.3 Safe clique partitions in trees

Now we consider the safe clique partition problem for trees. First we prove some lemmas about the nature of clique partitions for trees.

Lemma 4.6 *A clique partition of a tree with $n \geq 5$ can have at most one unsafe vertex.*

Proof. Suppose that a tree is partitioned into cliques $V_1, V_2, ..., V_q$ and that there are two unsafe vertices. Since $n \geq 5$ and a clique has at most 2 vertices, there are at least 3 cliques. If the hypothesized unsafe vertices are in the same clique, they must both be adjacent to the third clique. But this induces a cycle in G which cannot exist in a tree. If the unsafe vertices are in different cliques a cycle is again present; one path goes directly bewteen these two cliques and the other goes between them via the third clique. Therefore there can be only one unsafe vertex. □

Corollary 4.7 *If a clique partition of a tree has an unsafe vertex, it is in a K_2, not a K_1, in the partition.*

To simplify what follows, we will call a matching M in a tree T *safe* if it corresponds to a safe clique partition of T, and *unsafe* otherwise.

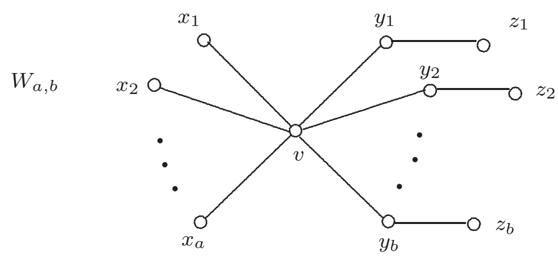

Figure 3: The graph $W_{a,b}$ in Lemma 4.8

Lemma 4.8 *Let G be a tree of order $n \geq 5$. A maximum matching M^* of G is unsafe if and only if G is the wounded spider $W_{a,b}$ $(a \geq 1, b \geq 0)$ in Figure 3 below.*

Proof. Let M^* be any unsafe maximum matching of G. By Lemma 4.6 there is a unique vertex $v \in V(G)$ such that for all i $(1 \leq i \leq q)$, $N(v) \cap V_i \neq \emptyset$ in the clique partition $V_1, V_2, ..., V_q$ corresponding to M^* in G. Since G is a tree, it follows that $V_i \cong K_1$ or $V_i \cong K_2$ for all i $(1 \leq i \leq q)$. Also, v is adjacent to copies of K_1 and K_2 only, and to all of them. Since $v \in V_j$, for some j $(1 \leq j \leq q)$, it follows that $V_j \cong K_2$, for otherwise $N(v) \cap V_j = \emptyset$. Thus $G \cong W_{a,b}$.

On the other hand, let $G \cong W_{a,b}$. Then M^* is the set of edges $\{(y_k, z_k) : 1 \leq k \leq b\} \cup \{(v, x_t) : \exists t, 1 \leq t \leq a\}$. Then M^* is unsafe because for all i $N(v) \cap V_i \neq 0$. $\square$

Corollary 4.9 *Let G be a tree of order $n \geq 5$, and M^* be a maximum matching in G. If M^* is safe, then any other maximum matching of G is safe. Thus the maximum matchings in a tree of order $n \geq 5$ are all safe or all unsafe.*

5 Discussion

The results of the previous section lead to an efficient algorithm for computing $\overline{\chi_d}(G)$ when G is a tree. Simply find a maximum matching M^*, and consider the clique partition $V_1, V_2, ..., V_q$ it corresponds to. If it is safe, then $\overline{\chi_d}(G) = n - |M^*|$ following Observation 4.3. If not, we know from Lemma 4.6 that there is only one unsafe vertex; suppose it is in clique V_i. Following Corollary 4.7, we know $V_i \cong K_2$. We now create a new clique partition by replacing V_i with two copies of K_1. This new clique partition is safe since the previously unsafe vertex is now in a K_1 in the partition and must be safe (recall the last sentence of Section 4.1). The new clique partition has exactly one more clique than did the previous one, and this is a smallest possible safe clique partition in this case. Therefore, when M^* is unsafe, $\overline{\chi_d}(G) = n - |M^*| + 1$. By Lemma 4.8 we can be sure that no other maximum matching can lead to a smaller clique partition. The complexity of the algorithm is that of maximum matching on trees.

We also note that this procedure can be used to find the dominator chromatic number $\chi_d(G)$ when $\overline{G}$ is a tree. In some cases, we can compute $\chi_d(G)$ even when $\overline{G}$ is *not* a tree, as the following example demonstrates.

Consider the Petersen graph and its complement, shown in Figure 4. In this section, we refer to these graphs as P and $\overline{P}$, respectively. Even though it is not a tree, P is triangle free. Accordingly, we observe that the matching $M = \{\{1,6\},\{2,7\},\{3,8\},\{4,9\},\{5,10\}\}$ is maximum and also safe. Therefore, by Observation 4.3, we know $\overline{\chi}_d(P) = n - |M^*| = 5$. In addition, the set M is a safe clique partition satisfying $\overline{\chi}_d(P) = 5$. Finally, M also gives a minimum dominator coloring of $\overline{P}$ and therefore $\chi_d(\overline{P}) = 5$.

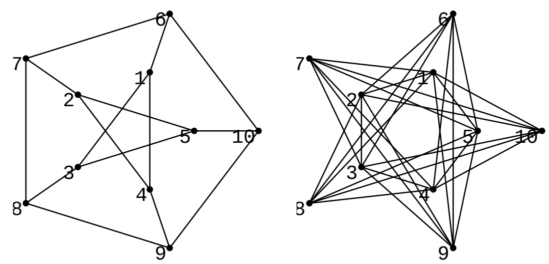

Figure 4: The Petersen Graph and its Complement

This note leaves several questions unanswered and raises some new ones. Although $\gamma(G)$, $\chi(G)$, and $\overline{\chi}_d(G)$ (described herein) can be computed in polynomial time when G is a tree, the status of computing $\chi_d(G)$ remains open for trees. Even an efficient algorithm for computing $\chi_d(G)$ when G is an arbitrary caterpillar seems a worthwhile contribution.

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